Semigroup Forum (2011) 83:313–319
DOI 10.1007/s00233-011-9311-0
RESEARCH ARTICLE

On regularity of sup-preserving maps: generalizing Zareckiĭ's theorem

Ulrich Höhle · Tomasz Kubiak

Received: 7 March 2011 / Accepted: 13 May 2011 / Published online: 8 June 2011

Abstract A sup-preserving map f between complete lattices L and M is regular if there exists a sup-preserving map g from M to L such that $fgf = f$. In the class of completely distributive lattices, this paper demonstrates a necessary and sufficient condition for f to be regular. When $L = M$ is a power set, our theorem reduces to the well known Zareckiĭ's theorem which characterizes regular elements in the semigroup of all binary relations on a set. Another application of our result is a generalization of Zareckiĭ's theorem for quantale-valued relations.

Keywords Sup-preserving map · Regular map · Complete distributivity · Regular relation · Quantale · Quantale-valued relation

1 Introduction

A morphism $A \xrightarrow{f} B$ in a category **C** is *regular* if there is a morphism $B \xrightarrow{g} A$ in **C** with $fgf = f$ (cf. [4]).

The purpose of this paper is to examine regular morphisms in the category **Sup** whose objects are all complete lattices and whose morphisms are all sup-preserving

Communicated by Boris M. Schein.

The second named author gratefully acknowledges the grant MTM2009-12872-C02-02 from the Ministry of Science and Innovation of Spain.

U. Höhle
Fachbereich C Mathematik und Naturwissenschaften, Bergische Universität, Gaußstraße 20, 42097 Wuppertal, Germany
e-mail: uhoehle@uni-wuppertal.de

T. Kubiak (✉)
Wydział Matematyki i Informatyki, Uniwersytet im. Adama Mickiewicza, Umultowska 87, 61-614 Poznań, Poland
e-mail: tkubiak@amu.edu.pl

maps (cf. [3]). We give a sufficient condition for a morphism of **Sup** to be regular as well as a necessary one. Both these conditions involve complete distributivity, so that—when gathered together—they provide a characterization of regular morphisms of the category of all completely distributive lattices and their sup-preserving maps (see Theorem 3.1). Our result can be viewed as a non-atomic generalization of the famous Zareckiĭ's theorem which characterizes regular elements in the semigroup of all binary relations on a set (cf. [14]). As a matter of fact, we generalize a recent extension of Zareckiĭ's theorem stated in [12] which characterizes regular morphisms in the category of all relations. We recall that $\rho \subseteq X \times Y$ is called *regular* if there exists $\sigma \subseteq Y \times X$ such that $\rho\sigma\rho = \rho$. The result of [12] says: *A relation* $\rho \subseteq X \times Y$ *is regular iff the lattice* $\{\rho(A) \mid A \subseteq X\}$ *is completely distributive* (see Sect. 2 for notation if needed and cf. Note 1.2 below). In the case $X = Y$ we have the just mentioned Zareckiĭ's theorem. It may be remarked that with an idempotent $\rho \subseteq X \times X$ (i.e. $\rho\rho = \rho$), the Zareckiĭ's theorem is already seen in [9].

Another application of Theorem 3.1 is a generalization of Zareckiĭ's theorem for quantale-valued relations (see Theorem 4.3).

The extensive literature related to Zareckiĭ's theorem include [1, 5, 8, 11–13] among others. In particular, alternative proofs are given in [1, 11, 13] all of which use the fact that $(\mathbf{2}^X, \subseteq)$ is an atomic completely distributive lattice ($\mathbf{2}^X$ is the power set of X).

Note 1.1 When saying that a morphism f of **Sup** is regular with $fgf = f$, we of course mean that g is a morphism of **Sup** too. We recall that in **Set** each map is regular (cf. [4]).

Note 1.2 Let $L \xrightarrow{f} M$ be a morphism in **Sup**. Then the range $f(L)$ is a complete lattice w.r.t. the partial ordering inherited from M and sups in $f(L)$ are formed in M, but in general not infs. Obviously, $f(L)$ together with the inclusion map is the image of f w.r.t. the (extremal epi, mono)-factorization property in **Sup**. In this context, saying that $f(L)$ is completely distributive simply means that the complete lattice $f(L)$ is completely distributive.

2 Binary relations as sup-preserving maps

Consider the category **Rel** whose objects are sets and $\rho : X \rightharpoondown Y$ is a morphism if $\rho \subseteq X \times Y$. The composition of $\rho \subseteq X \times Y$ and $\sigma \subseteq Y \times Z$ is the relation $\sigma\rho \subseteq X \times Z$ where

$$\sigma\rho = \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in \rho \text{ and } (y, z) \in \sigma\}.$$

Given $\rho \subseteq X \times Y$ and $A \subseteq X$, let $\rho(A)$ be defined by $\rho(A) = \bigcup_{x \in A} \rho(x)$ where $\rho(x) = \{y \in Y \mid (x, y) \in \rho\}$. The assignment $A \longmapsto \rho(A)$ preserves arbitrary unions, hence $\{\rho(A) \mid A \subseteq X\}$ is a complete lattice w.r.t the partial ordering inherited by $\mathbf{2}^Y$. Thus, each $\rho \subseteq X \times Y$ determines a union-preserving map $\mathbf{2}^X \xrightarrow{\Phi(\rho)} \mathbf{2}^Y$ by

$$\Phi(\rho)(A) = \rho(A).$$

Conversely (cf. [2, 6]), any map $\mathbf{2}^X \xrightarrow{f} \mathbf{2}^Y$ determines $\Sigma(f) \subseteq X \times Y$ by

$$(x, y) \in \Sigma(f) \Leftrightarrow y \in f(\{x\}).$$

If f is union-preserving, then Φ sends $\Sigma(f)$ back to f (cf. property (2) below). The category **Rel** is isomorphic to a full subcategory of **Sup**. In fact, the following hold:

Proposition 2.1 *Let $\rho \subseteq X \times Y$ and $\sigma \subseteq Y \times Z$ be relations, let $\mathbf{2}^X \xrightarrow{f} \mathbf{2}^Y \xrightarrow{g} \mathbf{2}^Z$ be arbitrary maps. Then*:

(1) $\Phi(\sigma\rho) = \Phi(\sigma)\Phi(\rho)$ *and* $\Sigma(gf) = \Sigma(g)\Sigma(f)$,
(2) $\Sigma(\Phi(\rho)) = \rho$, *and* $\Phi(\Sigma(f)) = f$ *provided f is union-preserving.*

From Proposition 2.1 we immediately obtain:

Fact 2.2 *The following statements hold*:

(1) *A relation $\rho \subseteq X \times Y$ is regular iff $\mathbf{2}^X \xrightarrow{\Phi(\rho)} \mathbf{2}^Y$ is regular.*
(2) *A union-preserving map $\mathbf{2}^X \xrightarrow{f} \mathbf{2}^Y$ is regular iff $\Sigma(f) \subseteq X \times Y$ is a regular relation.*

The Zareckiĭ's theorem (in its more general version of [12]) can now be formulated as follows:

Theorem 2.3 *A union-preserving map $\mathbf{2}^X \xrightarrow{f} \mathbf{2}^Y$ is regular iff $f(\mathbf{2}^X)$ is a completely distributive lattice.*

This formulation is an invitation to consider it in a more general lattice-theoretic setting by replacing the power sets by completely distributive lattices (see Theorem 3.1). We recall that the proofs of Zareckiĭ's theorem given in [1, 11, 13] make an essential use of the fact that $\mathbf{2}^X$ is an *atomic* completely distributive lattice. The proof of Theorem 3.1 is valid for arbitrary completely distributive lattices and is thus different from all of them.

3 Regular and sup-preserving maps

It is well known that a complete lattice L is completely distributive iff L^{op} is. In particular, for each family $\{A_i\}_{i\in I}$ of subsets of L the following holds:

$$\bigvee_{i\in I} \bigwedge A_i = \bigwedge_{\varphi \in \prod_{i\in I} A_i} \Big(\bigvee_{i\in I} \varphi(i) \Big).$$

Given a complete lattice L, we write $a \lhd b$ iff, whenever $C \subseteq L$ and $b \leq \bigvee C$, there is $c \in C$ with $a \leq c$ (cf. [3, 9]). Then L is completely distributive iff $\lhd$ has the approximation property:

$$a = \bigvee \{b \in L \mid b \lhd a\}$$

for each $a \in L$. For all $a, b, c, d \in L$ the following hold (cf. [3]):

($\lhd_1$) $a \lhd b$ implies $a \le b$,
($\lhd_2$) $c \le a \lhd b \le d$ implies $c \lhd d$,
($\lhd_3$) $a \lhd b$ implies $a \lhd c \lhd b$ for some $c \in L$.

The following is our generalization of the Zareckiĭ's theorem.

Theorem 3.1 *Let L and M be completely distributive lattices, and let $L \xrightarrow{f} M$ be sup-preserving. Then f is regular iff $f(L)$ is a completely distributive lattice.*

Since both the *if* part and the *only if* part of Theorem 3.1 hold true in a more general setting, we wish to split the theorem into two propositions from which Theorems 3.1 follows immediately. In fact, the *only if* part is also valid for L a continuous lattice (cf. Remark 3.4).

Proposition 3.2 *Let L and M be complete lattices, and let $L \xrightarrow{f} M$ be sup-preserving. If both M and $f(L)$ are completely distributive, then f is regular.*

Proof For every $y \in M$ we define $A_y = \{c \in L \mid y \le f(c)\}$. Then we put

$$G_y = \Big\{ b \in L \mid f(b) = \bigwedge f(A_y) \Big\}$$

where $\bigwedge$ is performed in $f(L)$. Define a map $M \xrightarrow{g} L$ by

$$g(x) = \bigvee_{y \lhd x} \bigvee G_y$$

for all $x \in M$. Let $A \subseteq M$. By using ($\lhd_1$)–($\lhd_3$), we have $y \lhd \bigvee A$ iff $y \lhd x$ for some $x \in A$ so that $g(\bigvee A) = \bigvee(\bigcup_{x \in A} \bigcup_{y \lhd x} G_y) = \bigvee_{x \in A} g(x)$, hence g is sup-preserving.

In order to verify $fgf = f$ we proceed as follows. As f is sup-preserving we have $\bigwedge f(A_y) = f(\bigvee G_y)$ and subsequently

$$\bigvee_{y \lhd f(a)} \bigwedge f(A_y) = f\Bigg(\bigvee_{y \lhd f(a)} \bigvee G_y \Bigg) = fgf(a), \quad a \in L.$$

Now we make use of the complete distributivity of $f(L)$ and obtain:

$$\bigwedge_{\varphi \in \prod\limits_{y \lhd f(a)} f(A_y)} \Bigg(\bigvee_{y \lhd f(a)} \varphi(y) \Bigg) = fgf(a).$$

Because of $\varphi(y) \in f(A_y)$ we have $y \le \varphi(y)$, and the relation

$$f(a) = \bigvee_{y \lhd f(a)} y \le fgf(a)$$

follows. On the other hand $y \lhd f(a)$ implies $y \leq f(a)$, and consequently $f(a) \in f(A_y)$. Because of $\bigwedge f(A_y) \leq f(a)$ the relation $fgf(a) \leq f(a)$ holds. □

Proposition 3.3 *Let L and M be complete lattices, let $L \xrightarrow{f} M$ be sup-preserving. If f is regular and L is completely distributive, then $f(L)$ is completely distributive too.*

Proof Let $fgf = f$ where $M \xrightarrow{g} L$ is sup-preserving, and let $a \in L$. Then $gf(a) \in L$ and by complete distributivity of L we have

$$gf(a) = \bigvee \{b \in L \mid b \lhd gf(a)\}.$$

We now check that $b \lhd gf(a)$ in L implies $f(b) \lhd f(a)$ in $f(L)$. Indeed, if $f(a) \leq \bigvee f(C)$ for some $C \subseteq L$, then $gf(a) \leq \bigvee gf(C)$ and there is a $c \in C$ such that $b \leq gf(c)$. This yields $f(b) \leq f(c)$, so that $f(b) \lhd f(a)$. Consequently,

$$\begin{aligned} f(a) &= fgf(a) \\ &\leq \bigvee \{f(b) \mid f(b) \lhd f(a)\} \\ &\leq \bigvee \{y \in f(L) \mid y \lhd f(a)\} \\ &\leq f(a). \end{aligned}$$

We have proved that $\lhd$ has the approximation property in $f(L)$. □

Remark 3.4 We recall that a complete lattice L is *continuous* if $a = \bigvee \{b \in L \mid b \ll a\}$ for all $a \in L$, where $b \ll a$ iff, whenever $a \leq \bigvee A$ for some $A \subseteq L$, there exists a finite subset $D \subseteq A$ such that $b \leq \bigvee D$. Every completely distributive lattice is continuous (see [3]). It is not difficult to observe that Proposition 3.3 maintains its validity for the relation $\ll$ and continuous lattices.

4 A quantalic version of Zareckiĭ's theorem

In what follows we extend our previous considerations in Sect. 2 from the two-valued to the many-valued setting. More specifically, we will replace maps $\mathbf{2}^X \to \mathbf{2}^Y$ by maps $L^X \to L^Y$ where the complete lattice L replaces the two point lattice $\mathbf{2}$. Here and elsewhere L^X is the set of all maps from X into L ordered pointwisely.

For this purpose we first recall that a triple $Q = (Q, \leq, \&)$ is called a *quantale* if $(Q, \leq)$ is a complete lattice, $(Q, \&)$ is a semigroup, and & distributes over arbitrary sups in both variables (cf. [10]). We begin with the following example (cf. [6, 7]).

Example 4.1 (*Quantale $\mathcal{Q}(L)$*) Let L be a complete lattice. On the set $\mathcal{Q}(L)$ of all sup-preserving self-maps on L we define a partial ordering $\leq$ in a pointwise way. Then $(\mathcal{Q}(L), \leq)$ is again a complete lattice. In particular, the sup of $\mathcal{S} \subseteq \mathcal{Q}(L)$ is given by $(\bigvee \mathcal{S})(a) = \bigvee_{\sigma \in \mathcal{S}} \sigma(a)$ for all $a \in L$. The composition

$$(\sigma_1 \,\&\, \sigma_2)(a) = \sigma_2(\sigma_1(a)), \quad a \in L,$$

induces a semigroup operation & on $\mathcal{Q}(L)$ so that the resulting triple $\mathcal{Q}(L) = (\mathcal{Q}(L), \leq, \&)$ becomes a quantale (in fact, a unital quantale—a detail not needed for our purposes).

Let Q be an arbitrary quantale, and X and Y be sets. A Q-valued relation $R : X \rightharpoondown Y$ is a map $X \times Y \xrightarrow{R} Q$. The *composition* of Q-valued relations $R : X \rightharpoondown Y$ and $S : Y \rightharpoondown Z$ is defined for all $x \in X$ and $z \in Z$ by

$$(SR)(x, z) = \bigvee_{y \in Y} R(x, y) \,\&\, S(y, z).$$

When $L = \mathbf{2}$, the category of all $\mathcal{Q}(L)$-valued relations can be identified with **Rel** because $\mathcal{Q}(\mathbf{2})$ is just the two-point lattice.

Now we observe that each $\mathcal{Q}(L)$-valued relation $R : X \rightharpoondown Y$ determines a sup-preserving map $L^X \xrightarrow{\Phi(R)} L^Y$ defined for all $f \in L^X$ and $y \in Y$ by

$$\Phi(R)(f)(y) = \bigvee_{x \in X} R(x, y)(f(x)).$$

On the other hand, each sup-preserving map $L^X \xrightarrow{\varphi} L^Y$ induces a $\mathcal{Q}(L)$-valued relation $X \times Y \xrightarrow{\Sigma(\varphi)} \mathcal{Q}(L)$ defined for all $a \in L, x \in X$, and $y \in Y$ by

$$\Sigma(\varphi)(x, y)(a) = \varphi(a1_x)(y),$$

where

$$a1_x(z) = \begin{cases} a, & z = x, \\ \perp, & z \neq x \end{cases}$$

and $\perp$ is the bottom element of L. The following provides a $\mathcal{Q}(L)$-valued counterpart (in fact, a generalization from $\mathbf{2}$ to L) of Proposition 2.1.

Proposition 4.2 *Let $R : X \rightharpoondown Y$ and $S : Y \rightharpoondown Z$ be $\mathcal{Q}(L)$-valued relations, and let $L^X \xrightarrow{\varphi} L^Y \xrightarrow{\psi} L^Z$ be arbitrary maps. Then:*

(1) $\Phi(SR) = \Phi(S)\Phi(R)$ *and* $\Sigma(\psi\varphi) = \Sigma(\psi)\Sigma(\varphi)$,
(2) $\Sigma(\Phi(R)) = R$, *and* $\Phi(\Sigma(\varphi)) = \varphi$ *provided φ is sup-preserving.*

Proof The proofs of (1) and the first part of (2) follows immediately from the definitions of all the involved compositions. Proving the second part of (2) requires a simple observation that each $f \in L^X$ has the following decomposition: $f = \bigvee_{x \in X} f(x)1_x$.

Indeed,

$$\begin{aligned}\Phi(\Sigma(\varphi))(f)(y) &= \bigvee_{x\in X} \varphi(f(x)1_x)(y)\\ &= \varphi\Big(\bigvee_{x\in X} f(x)1_x\Big)(y)\\ &= \varphi(f)(y)\end{aligned}$$

for all $f \in L^X$ and $y \in Y$. □

When L is completely distributive, then so is L^X. Using Theorem 3.1 and Proposition 4.2 (also cf. Note 1.2), we obtain our quantalic analogue of Zareckiĭ's theorem which when $L = \mathbf{2}$ yields the characterization of regularity of usual relations as given in Theorem 2.3.

Theorem 4.3 *Let L be a completely distributive lattice. Then a $\mathcal{Q}(L)$-valued relation $R : X \rightharpoondown Y$ is regular iff $\Phi(R)(L^X)$ is a completely distributive lattice.*

References

1. Bandelt, H.-J.: Regularity and complete distributivity. Semigroup Forum **19**, 123–126 (1980)
2. Erceg, M.A.: Functions, equivalence relations, quotient spaces and subsets in fuzzy set theory. Fuzzy Sets Syst. **3**, 75–92 (1980)
3. Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S.: A Compendium of Continuous Lattices. Springer, Berlin, Heidelberg, New York (1980)
4. Mac Lane, S.: Categories for the Working Mathematician. Springer, New York, Heidelberg, Berlin (1971)
5. Markowsky, G.: Idempotents and product representations with applications to semigroup of binary relations. Semigroup Forum **5**, 95–119 (1972)
6. Mulvey, C.J., Pelletier, J.W.: A quantisation of the calculus of relations. In: CMS Proceedings, vol. 13, pp. 345–360. Am. Math. Soc., Providence (1992)
7. Pelletier, J.W., Rosický, J.: Simple involutive quantales. J. Algebra **195**, 367–386 (1997)
8. Plemmons, R.J., West, M.T.: On the semigroup of binary relations. Pac. J. Math. **35**, 743–753 (1970)
9. Raney, G.N.: A subdirect-union representation of completely distributive lattices. Proc. Am. Math. Soc. **4**, 518–522 (1953)
10. Rosenthal, K.I.: Quantales and Their Applications. Pitman Research Notes in Mathematics, vol. 348. Longman Scientific & Technical, Longman House, Burnt Mill, Harlow (1996)
11. Schein, B.M.: Regular elements of the semigroup of all binary relations. Semigroup Forum **13**, 95–112 (1976)
12. Xu, X.-Q., Luo, M.-K.: Regular relations and normality of topologies. Semigroup Forum **72**, 477–480 (2006)
13. Yang, J.C.: A theorem on the semigroup of binary relations. Proc. Am. Math. Soc. **22**, 134–135 (1969)
14. Zareckiĭ, K.A.: The semigroup of binary relations. Mat. Sb. **61**, 291–305 (1963) (in Russian)